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On Submodularity of Quadratic Observation Selection in Constrained Networked Sensing Systems

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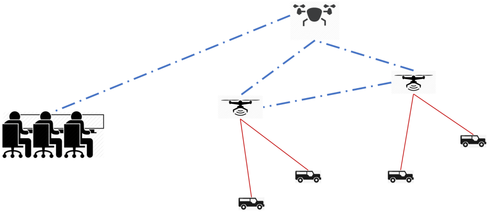
Introduction

- UAVs gathering range and angular measurements of targets' positions
- Estimation and tracking tasks in control unit
- Constraints due to communication cost, power consumption, computational burden



Goal

Communicate a subset of measurements to enable low mean square error (MSE) estimation and tracking of targets under constraints

$\mathbf{u}_k^i$: location of i^{th} UAV at time k $\mathbf{s}_k^j$: location of j^{th} object at time k

Measurement Model for Radar Systems

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- Range measurement

$$r_{ij} = \frac{1}{2}\|\mathbf{u}_k^i - \mathbf{s}_k^j\|_2^2 + \nu_{ij}$$

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- Angular measurements

$$\phi_{ij} = \arcsin \frac{u_k^i(3) - s_k^j(3)}{\|\mathbf{u}_k^i - \mathbf{s}_k^j\|_2} + \zeta_{ij}$$

$$\alpha_{ij} = \arctan \frac{u_k^i(1) - s_k^j(1)}{u_k^i(2) - s_k^j(2)} + \eta_{ij}$$

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$\longrightarrow$ Nonlinear measurement model

- Main challenge: We don't have an MVUE! $\longrightarrow$ Unknown Moment
- Locally-optimal observation selection [Flaherty'06, Krause'08]: linearize around a guess $\mathbf{x}_0$

$$\hat{y}_i = y_i - g_i(\mathbf{x}_0) \approx \nabla g_i(\mathbf{x}_0)^\top \mathbf{x} + v_i,$$

and find an approximate moment:

$$\hat{\mathbf{P}}_{\mathcal{S}} = \left(\boldsymbol{\Sigma}_x^{-1} + \sum_{i \in \mathcal{S}} \frac{1}{\sigma_i^2} \nabla g_i(\mathbf{x}_0) \nabla g_i(\mathbf{x}_0)^\top \right)^{-1}$$

Challenges of Nonlinearity

- Main challenge: We don't have an MVUE! $\longrightarrow$ Unknown Moment
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- **Observation selection task:** Minimize a scalarization of $\hat{\mathbf{P}}_{\mathcal{S}}$ subject to cardinality constraint

Proposed Approach

Van Trees Bound (VTB) as Selection Metric

- Objective from linearized model has no known connection to MSE

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Main Idea

Exploiting Van Trees' bound (VTB) on error covariance of weakly biased estimators

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Main Idea

Exploiting Van Trees' bound (VTB) on error covariance of weakly biased estimators

- Class of quadratic measurement models
- Contribution:
  - Deriving new optimality criteria by relying on Van Trees' inequality
  - Proving special properties of the proposed criteria
  - Developing a greedy selection algorithm with theoretical bounds on its achievable utility

Theorem [Van Trees'1968]

Let $\mathbf{x}$ be a collection of random unknown parameters, and let $\mathbf{y}_{\mathcal{S}} = \{y_i\}_{i \in \mathcal{S}}$ denote the collection of measurements indexed by the subset $\mathcal{S}$. For any estimator $\hat{\mathbf{x}}_{\mathcal{S}}$ that satisfies

$$\int_{-\infty}^{+\infty} \nabla_{\bar{\mathbf{x}}} \left(p_{\mathbf{x}}(\bar{\mathbf{x}}) \, \mathbb{E}_{\mathbf{y}|\mathbf{x}}[\hat{\mathbf{x}}_{\mathcal{S}} - \bar{\mathbf{x}}] \right) d\bar{\mathbf{x}} = \mathbf{0},$$

it holds that

$$\mathbf{P}_{\mathcal{S}} \succeq \mathbb{E}_{\mathbf{y}_{\mathcal{S}},\mathbf{x}} \left[(\nabla_{\bar{\mathbf{x}}} \log q_{\mathbf{x}}(\bar{\mathbf{x}}))(\nabla_{\bar{\mathbf{x}}} \log q_{\mathbf{x}}(\bar{\mathbf{x}}))^{\top} \right]^{-1},$$

where $q_{\mathbf{x}}(\bar{\mathbf{x}}) = p_{\mathbf{y}_{\mathcal{S}},\mathbf{x}}(\bar{\mathbf{x}}, ; \mathbf{y})$ is the posterior distribution of $\mathbf{x}$ given $\mathbf{y}_{\mathcal{S}}$.

- Quadratic relation between observations and unknown parameters

$$\mathbf{y}_i = \underbrace{\frac{1}{2}\mathbf{x}^\top \mathbf{Z}_i \mathbf{x} + \mathbf{h}_i^\top \mathbf{x}}_{g_i(\mathbf{x})} + \mathbf{v}_i \ , \quad i \in \{1, 2, \dots, n\}$$

- A closed-form expression for VTB of quadratic models

Theorem

For any weakly biased estimator $\hat{\mathbf{x}}_{\mathcal{S}}$ with error covariance $\mathbf{P}_{\mathcal{S}}$ it holds that

$$\mathbf{P}_{\mathcal{S}} \succeq \left(\sum_{i \in \mathcal{S}} \frac{1}{\sigma_i^2} \left(\mathbf{Z}_i \boldsymbol{\Sigma}_x \mathbf{Z}_i^\top + \mathbf{h}_i \mathbf{h}_i^\top \right) + \mathbf{I}_x \right)^{-1} = \mathbf{B}_{\mathcal{S}}$$

- Scalarizations of VTB $\mathbf{B}_{\mathcal{S}}$ as objective functions
    - logdet(.) scalarization: $f^{D}(\mathcal{S}) := \operatorname{logdet}\left(\mathbf{B}_{\mathcal{S}}^{-1}\right) - \operatorname{logdet}\left(\mathbf{I}_x\right)$
    - Tr(.) scalarization: $f^{A}(\mathcal{S}) := \operatorname{Tr}(\mathbf{I}_x^{-1} - \mathbf{B}_{\mathcal{S}})$

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- Optimization formulation

logdet formulation

$$\begin{aligned} \underset{\mathcal{S}}{\text{maximize}} \quad & f^D(\mathcal{S}) \\ \text{s.t.} \quad & \mathcal{S} \subset [n], \quad |\mathcal{S}| = K \end{aligned}$$

Trace formulation

$$\begin{aligned} \underset{\mathcal{S}}{\text{maximize}} \quad & f^A(\mathcal{S}) \\ \text{s.t.} \quad & \mathcal{S} \subset [n], \quad |\mathcal{S}| = K \end{aligned}$$

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- An NP-hard, combinatorial problem [Natarajan'95] $\rightarrow$ resort to approximation methods

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  - Initialize $\mathcal{S}^g = \emptyset$
  - For K iterations:
    - Find sensor $j_s \in \mathcal{X} \backslash \mathcal{S}^g$ with the largest marginal gain
    - Update current selection: $\mathcal{S}^g \leftarrow \mathcal{S}^g \cup \{j_s\}$

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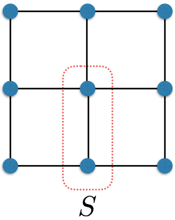
- Polynomial complexity in the number of oracle calls

Theoretical Results

- Set function: A function that assigns a value to each subset of a ground set $\mathcal{X}$

  Example: Value of a cut $f(\mathcal{S})$ for all $\mathcal{S} \subseteq \mathcal{V}$ in an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

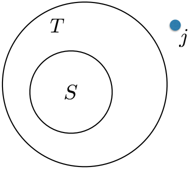


- Monotonicity: $f(\mathcal{S}) \leq f(\mathcal{T})$ for all $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{X}$

(Weak) Submodularity

- Marginal gain: $f_j(\mathcal{S}) = f(\mathcal{S} \cup \{j\}) - f(\mathcal{S})$

  Gain we get by adding j to $\mathcal{S}$



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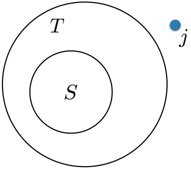

- Submodularity: $f_j(\mathcal{S}) \geq f_j(\mathcal{T})$ for all $\mathcal{S} \subseteq \mathcal{T} \subset \mathcal{X}$ and $j \in \mathcal{X} \backslash \mathcal{T}$
  Diminishing returns property
- α_f-Weak Submodularity: $\alpha_f \times f_j(\mathcal{S}) \geq f_j(\mathcal{T})$ where $\alpha_f > 1$ for all combinations of $(\mathcal{S}, \mathcal{T}, j)$

Characterizing logdet Scalarization

- Greedy maximization performance for normalized, monotone, and weak submodular functions [Nemhauser'78]:

$$f(\mathcal{S}) \geq (1 - e^{-\frac{1}{\alpha_f}}) f(\mathcal{O})$$

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Theorem

- $f^D(\emptyset) = 0$ (normalized)
- $f^D(\mathcal{S})$ is monotone (higher values as we keep selecting more observations)
- $f^D(\mathcal{S})$ is submodular (i.e. $\alpha_{f^D} \leq 1$)

Theorem

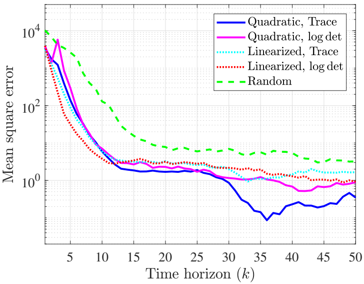
- $f^A(\emptyset) = 0$ (normalized)
- $f^A(\mathcal{S})$ is monotone (higher values as we keep selecting more observations)
- If $\mathbf{h}_i = \mathbf{0}$ and $\mathbf{Z}_i = \mathbf{z}_i \mathbf{z}_i^\top$, then

$$\alpha_{f^A} \leq \max_{j \in [n]} \frac{\lambda_{\max}(\boldsymbol{\Sigma}_x)^2 (\lambda_{\max}(\sigma_j^2 \boldsymbol{\Sigma}_x) + 1)}{\lambda_{\min}(\mathbf{B}_{[n]})^2 (\lambda_{\min}(\sigma_j^2 \boldsymbol{\Sigma}_x) + 1)}.$$

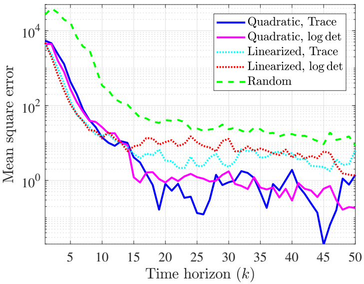
- Interpretation as SNR condition

Simulation Results

- Tracking by extended Kalman filter over a priod of 100 time steps
- 20 targets, 20 UAVs, around 600 distance and angular measurements, selecting $K = 100$

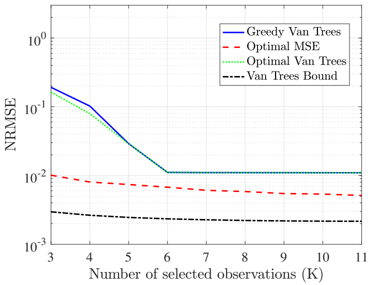


(a) Identical noise powers

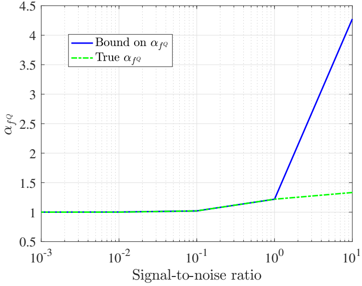


(b) Random noise powers

Evaluation of Theoretical Results



(c) Tightness of VTB



(d) Bound on α_{fA}

- Asymptotic tightness of VTB
- Tightness of weak submodularity bound in low SNR regime

Conclusion

Summary and Future Work

Summary:

- Utilizing Van Trees' inequality to derive new optimality criteria for quadratic observation models
- Showed monotonicity and (weak) submodularity of $\log\det$ and trace scalarizar,itions
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- Showed monotonicity and (weak) submodularity of $\log\det$ and trace scalarizarititions
- Analyzing the performance of a proposed greedy maximization algorithm

Future Work:

- Analyzing the performance of VTB-based criteria in second-order approximation of general nonlinear observation models

Thank you!

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